Section 4.7 Cauchy-Euler Equations

Format: $a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = g(x)$

$x^n \rightarrow$ matches n^{th} derivative

Example $3x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 5y = 0$

These equations are solved in a similar manner to linear equations with constant coefficients.

Let's develop the auxiliary equation:

starting point $y = x^m$

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$$y = x^m$$
$$y' = mx^{m-1}$$
$$y'' = m(m-1)x^{m-2}$$

$$ax^2(m(m-1)x^{m-2}) + bx(mx^{m-1}) + cx^m = 0$$
$$\underbrace{ax^2x^{m-2}}\, m(m-1) + bx^1x^{m-1} \cdot m + cx^m = 0$$

$$a(m)(m-1)x^m + bmx^m + cx^m = 0$$

$$x^m(a(m)(m-1) + bm + c) = 0$$

$$x^m(am^2 - am + bm + c) = 0$$

$$(am^2 + (b-a)m + c)x^m = 0$$

$$\Rightarrow (am^2 + (b-a)m + c) = 0$$ This is not as direct as linear constant coefficients

$$\boxed{\text{Cauchy Euler Auxiliary Equation} \quad am^2 + (b-a)m + c = 0}$$

You can either memorize this equation or do a direct substitution for each problem. If the power is higher than 2, you will need to use a direct substitution of

$y = x^m$
$y' = mx^{m-1}$
$y'' = m(m-1)x^{m-2}$
etc.

3 types of solutions

option 1 2 distinct real roots. m_1, m_2 solve the auxiliary equation

$$\boxed{y_c = c_1 x^{m_1} + c_2 x^{m_2}}$$

Option 2 Repeat real root

Let's determine, using reduction of order, what the second solution should be.

auxiliary equation $am^2 + (b-a)m + c = 0$

$$m = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4(a)(c)}}{2a}$$

with $(b-a)^2 - 4ac = 0$

$$m_1 = m_2 = \frac{-(b-a)}{2a}$$

$$\boxed{\begin{aligned} y_1 &= x^{m_1} \\ y_1 &= x^{\frac{-(b-a)}{2a}} \end{aligned}}$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$$

$$\boxed{ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0}$$

convert to standard form to find $p(x)$

$$\frac{d^2y}{dx^2} + \frac{bx}{ax^2}\frac{dy}{dx} + \frac{c}{ax^2}y = 0$$

$$\frac{d^2y}{dx^2} + \underbrace{\frac{b}{ax}}_{\boxed{p(x)}}\frac{dy}{dx} + \underbrace{\frac{c}{ax^2}}_{Q(x)}y = 0$$

$$y_2 = x^{m_1} \int \frac{e^{-\int \frac{b}{ax}dx}}{(x^{m_1})^2}\,dx$$

$$y_2 = x^{m_1} \int \frac{\frac{b}{a}\ln x}{(x^{m_1})^2}\,dx$$

$$y_2 = x^{m_1} \int \frac{e^{-\frac{b}{a}\ln x}}{x^{2m_1}}\, dx$$

$$y_2 = x^{m_1} \int \frac{e^{\ln x^{-b/a}}}{x^{2m_1}}\, dx$$

$$y_2 = x^{m_1} \int \frac{x^{-b/a}}{x^{2m_1}}\, dx$$

$$y_2 = x^{m_1} \int x^{-b/a}\, x^{-2m_1}\, dx$$

$$y_2 = x^{m_1} \int x^{-b/a}\, x^{-2\left(-\frac{(b-a)}{2a}\right)}\, dx$$

$$y_2 = x^{m_1} \int x^{-b/a}\, x^{\frac{b-a}{a}}\, dx$$

$$y_2 = x^{m_1} \int x^{-b/a}\, x^{\frac{b}{a}}\, x^{-1}\, dx$$

$$y_2 = x^{m_1} \int \frac{1}{x}\, dx$$

$$y_2 = x^{m_1} \cdot \ln x$$

So to adjust a repeated solution, multiply by $\ln x$

Summary of the 3 cases

case 1 2 distinct real roots m_1, m_2

$$\boxed{y_c = c_1 x^{m_1} + c_2 x^{m_2}}$$

case 2 Repeated real root

$$\boxed{y_c = c_1 x^{m_1} + c_2 x^{m_1} \ln x}$$

It is okay to drop $|\,|$ on $\ln|x|$

case 3 2 complex roots

$$\boxed{y_c = c_1 x^{\alpha + \beta i} + c_2 x^{\alpha - \beta i}}$$

or

$$\boxed{y_c = x^{\alpha}\left(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right)}$$

Example 1 Distinct Roots

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

$y = x^m$
$y' = m x^{m-1}$
$y'' = m(m-1)x^{m-2}$

Option #1 $a=1 \quad b=-2 \quad c=-4$

$$am^2 + (b-a)m + c = 0$$
$$m^2 + (-2-1)m + -4 = 0$$
$$m^2 - 3m - 4 = 0$$
$$(m-4)(m+1) = 0$$

$m_1 = 4 \quad m_2 = -1$

$$\boxed{y_c = c_1 x^4 + c_2 x^{-1}}$$

$\boxed{\text{Option \#2}}$

$y = x^m$
$y' = mx^{m-1}$
$y'' = m(m-1)x^{m-2}$

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

$$x^2(m(m-1)x^{m-2}) - 2x(mx^{m-1}) - 4x^m = 0$$
$$x^m(m^2 - m) - 2mx^m - 4x^m = 0$$
$$x^m(m^2 - m - 2m - 4) = 0$$

$$m^2 - 3m - 4 = 0$$
$$(m-4)(m+1) = 0$$

$$\boxed{y_c = c_1 x^4 + c_2 x^{-1}}$$

$m_1 = 4 \quad m_2 = -1$

$\boxed{\text{Example 2}}$ Repeated Roots.

$$4x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = 0$$

$a = 4 \quad b = 8 \quad c = 1$
$$am^2 + (b-a)m + c = 0$$
$$4m^2 + (8-4)m + 1 = 0$$
$$4m^2 + 4m + 1 = 0$$
$$(2m+1)(2m+1) = 0$$

$$m_1 = m_2 = -\frac{1}{2}$$

$$\boxed{y_c = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x}$$

Example 3 Complex Solutions

Solve $4x^2y'' + 17y = 0$ $\quad y(1) = -1$, $\quad y'(1) = -\frac{1}{2}$

$a = 4 \quad b = 0 \quad c = 17$

$am^2 + (b-a)m + c = 0$

$4m^2 + (0-4)m + 17 = 0$

$4m^2 - 4m + 17 = 0$

$$\begin{array}{r|l} 2 & 120 \\ 2 & 60 \\ 2 & 30 \\ 3 & 15 \\ & 5 \end{array}$$

$$m = \frac{4 \pm \sqrt{16 - 4(4)(17)}}{2(4)} = \frac{4 \pm \sqrt{-256}}{8} =$$

$$m = \frac{4 \pm 16i}{8} = \frac{1}{2} \pm 2i \qquad \alpha = \frac{1}{2} \quad \beta = 2$$

<u>format</u>

$$y = x^{\alpha}\left(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right)$$

$$\boxed{y = x^{1/2}\left(c_1 \cos(2\ln x) + c_2 \sin(2 \ln x)\right)}$$

now we need c_1 & c_2

$$y(1) = 1^{1/2}\left(c_1 \cos(2\ln(1)) + c_2 \sin(2\ln(1))\right) = -1$$

$$y(1) = 1\left(c_1 \overset{1}{\cancel{\cos(0)}} + c_2 \overset{0}{\cancel{\sin(0)}}\right) = -1$$

$$c_1 = -1$$

product

$$y' = x^{1/2}\left(c_1(-\sin(2\ln x)) \cdot \frac{2}{x} + c_2(\cos(2\ln x)) \cdot \frac{2}{x}\right)$$

$$+ \frac{1}{2}x^{-1/2}\left(c_1(\cos(2\ln x)) + c_2 \sin(2\ln x)\right)$$

$$y'(1) = -\frac{1}{2} \rightarrow c_2 = 0$$

example/practice
Cauchy Euler

$\boxed{Ex\ 4}$ $$x^3 \frac{d^3y}{dx^3} + 5x^2 \frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y = 0$$

let $\boxed{y = x^m}$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$y''' = m(m-1)(m-2)x^{m-3}$

$$x^3(m(m-1)(m-2)(x^{m-3}) + 5x^2(m)(m-1)x^{m-2} + 7x(mx^{m-1}) + 8x^m = 0$$

$$m(m-1)(m-2)x^m + 5m(m-1)x^m + 7mx^m + 8x^m = 0$$

$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$

auxillary eqn.

$m(m^2-3m+2) + 5m^2 - 5m + 7m + 8 = 0$

$m^3 - 3m^2 + 2m + 5m^2 - 5m + 7m + 8 = 0$

$m^3 + 2m^2 + 4m + 8 = 0$

$m^2(m+2) + 4(m+2) = 0$

$(m^2+4)(m+2) = 0$ $\alpha = 0 \quad \beta = 2$

$\boxed{m = -2}$ $\boxed{m = \pm 2i}$ $\boxed{\text{complex}}$

$\downarrow$ $c_1 x^{-2}$ $\downarrow$ $x^\alpha(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x))$

or

$c_1 x^{\alpha+\beta i} + c_2 x^{\alpha-\beta i}$

$$y = c_1x^{-2} + x^0(c_2\cos(2\ln x) + c_3\sin(2\ln x))$$

$$\boxed{y = c_1x^{-2} + c_2\cos(2\ln x) + c_3\sin(2\ln x)}$$

When the equation is not homogeneous ($g(x) \neq 0$), you find y_p in the same manner as you did with linear equations with constant coefficients.

$\boxed{\text{option 1}}$ undetermined coeffients (when applicable)

$\boxed{\text{option 2}}$ Variation of parameters.

Recall the basic formula for variation of parameters.

$$\boxed{y_c = c_1 y_1 + c_2 y_2} \qquad \frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy = f(x)$$

1

(standard form)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad u_1' = \frac{W_1}{W} \rightarrow u_1 = \int \frac{W_1}{W}\,dx$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} \qquad u_2' = \frac{W_2}{W} \longrightarrow u_2 = \int \frac{W_2}{W}\,dx$$

<u>solution</u> $y = y_c + y_p = c_1y_1 + c_2y_2 + u_1y_1 + u_2y_2$

$\boxed{\text{Note}}$

Convert the Cauchy-Euler equations to standard form before using either undetermined coefficients or variation of parameters to find y_p.

Example 5 $\boxed{x^2y'' - 3xy' + 3y = 2x^4e^x}$

y_c

$am^2 + (b-a)m + c = 0$ $\quad a = 1$

$m^2 + (-3-1)m + 3 = 0$ $\quad b = -3$

$m^2 - 4m + 3 = 0$ $\quad c = 3$

$(m-3)(m-1) = 0$

$m_1 = 3$ $\quad m_2 = 1$

$$\boxed{y_c = c_1x^3 + c_2x}$$

y_p with variation of parameters

$$\frac{x^2y''}{x^2} - \frac{3xy'}{x^2} + \frac{3y}{x^2} = \frac{2x^4e^x}{x^2}$$

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

standard form

so $y_1 = x^3$ $\quad f(x) = 2x^2e^x$

$y_2 = x$

$$\boxed{W} = \begin{vmatrix} x^3 & x \\ 3x^2 & 1 \end{vmatrix} = x^3 - 3x^3 = \boxed{-2x^3}$$

$$\boxed{W_1} = \begin{vmatrix} 0 & x \\ 2x^2e^x & 1 \end{vmatrix} = 0 - 2x^3e^x = \boxed{-2x^3e^x}$$

$$\boxed{W_2} = \begin{vmatrix} x^3 & 0 \\ 3x^2 & 2x^2e^x \end{vmatrix} = 2x^5e^x - 0 = \boxed{2x^5e^x}$$

$$u_1' = \frac{W_1}{W} = \frac{-2x^3e^x}{-2x^3} = e^x$$

$$u_1 = \int e^x\,dx = e^x \qquad \boxed{u_1 = e^x}$$

$$u_2' = \frac{W_2}{W} = \frac{2x^5e^x}{-2x^3} = -x^2e^x$$

$$u_2 = \int -x^2e^x\,dx$$

	D	I
+	$-x^2$	e^x
−	$-2x$	e^x
+	-2	e^x
−	0	e^x

$\boxed{\text{by parts}}$

$$u_2 = -x^2e^x + 2xe^x - 2e^x + \cancel{C} \to 0$$

$$\boxed{u_2 = -x^2e^x + 2xe^x - 2e^x}$$

$$y = y_c + y_p$$

$$y = c_1x^3 + c_2x + e^x(x^3) + (-x^2e^x + 2xe^x - 2e^x)x$$

$$y = c_1x^3 + c_2x + \underbrace{\cancel{x^3e^x} - \cancel{x^3e^x}} + 2x^2e^x - 2xe^x$$

$$\boxed{y = c_1x^3 + c_2x + 2x^2e^x - 2xe^x}$$

12

option 2

undetermined coefficients

$$y_1 = x^3$$
$$y_2 = x$$

standard form $\boxed{y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 2x^2 e^x}$

$$y_p = (Ax^2 + Bx + C)e^x$$

$$y_p' = (Ax^2 + Bx + c)e^x + (2Ax + B)e^x$$
$$= (Ax^2 + Bx + c + 2Ax + B)e^x$$
$$= (Ax^2 + (B+2A)x + (B+c))e^x$$

$$y_p'' = (Ax^2 + (B+2A)x + (B+c))e^x + (2Ax + (B+2A))e^x$$

$$y_p'' = (Ax^2 + (B+2A)x + (B+c) + 2Ax + (B+2A))e^x$$
$$= (Ax^2 + (B+4A)x + (2A+2B+c))e^x$$

$$(Ax^2 + (B+4A)x + (2A+2B+c))e^x$$
$$-\frac{3}{x}(Ax^2 + (B+2A)x + (B+c))e^x$$
$$+\frac{3}{x^2}(Ax^2 + Bx + C)e^x = 2xe^x$$

$$(Ax^2 + (B+4A)x + (2A+2B+c))e^x$$
$$+\left(-3Ax - 3(B+2A) - \frac{3}{x}B - \frac{3}{x}C\right)e^x$$
$$+ e^x\left(+3A + \frac{3B}{x} + \frac{3C}{x^2}\right) = 2x^2e^x$$

$$x^2e^x(A) \quad + xe^x(B+4A-3A)$$
$$+ e^x(2A+2B+C-3(B+2A)+3A)$$
$$+ \frac{e^x}{x}(\cancel{-3B}-3C+\cancel{3B})$$
$$+ \frac{e^x}{x^2}(3C) = 2x^2e^x$$

$\boxed{A=2}$

$B+4A-3A=0$
$B+A=0$
$B=-A$
$\boxed{B=-2}$

$-3C=0$
$\boxed{C=0}$

$3C=0$
$\boxed{C=0}$

so $y_p = (Ax^2+Bx+C)e^x$

$$\boxed{y_p = 2x^2e^x - 2xe^x}$$

$$\boxed{y = y_c + y_p = C_1x^3 + C_2x + 2x^2e^x - 2xe^x}$$